Chapter 9

SEQUENCE AND SERIES

9.1 Overview

By a sequence, we mean an arrangement of numbers in a definite order according to some rule. We denote the terms of a sequence by $a_1, a_2, a_3, \ldots$, etc., the subscript denotes the position of the term.

In view of the above a sequence in the set X can be regarded as a mapping or a function $f : \mathbf{N} \to X$ defined by

$$f(n) = t_n \ \forall \ n \in \mathbf{N}.$$

Domain of f is a set of natural numbers or some subset of it denoting the position of term. If its range denoting the value of terms is a subset of **R** real numbers then it is called a **real sequence**.

A sequence is either finite or infinite depending upon the number of terms in a sequence. We should not expect that its terms will be necessarily given by a specific formula.

However, we expect a theoretical scheme or rule for generating the terms.

Let $a_1, a_2, a_3, \ldots$, be the sequence, then, the expression $a_1 + a_2 + a_3 + \ldots$ is called the **series** associated with given sequence. The series is finite or infinite according as the given sequence is finite or infinite.

Remark When the series is used, it refers to the indicated sum not to the sum itself. Sequence following certain patterns are more often called **progressions**. In progressions, we note that each term except the first progresses in a definite manner.

9.1.1 *Arithmetic progression (A.P.)* is a sequence in which each term except the first is obtained by adding a fixed number (positive or negative) to the preceding term.

Thus any sequence $a_1, a_2, a_3 \ldots a_n, \ldots$ is called an **arithmetic progression** if

$a_{n+1} = a_n + d, n \in \mathbf{N}$, where d is called the **common difference** of the A.P., usually we denote the first term of an A.P by a and the last term by l

The general term or the ***n*^th^ term of the A.P.** is given by

$$\boldsymbol{a_n = a + (n-1)\,d}$$

The n^{th} term from the last is given by $\quad a_n = l - (n-1)\,d$

The sum S_n of the first n terms of an A.P. is given by

$S_n = \frac{n}{2}\left[2a + (n-1)\,d\right] = \frac{n}{2}(a + l)$, where $l = a + (n-1)\,d$ is the last terms of the A.P.,

and the general term is given by $a_n = S_n - S_{n-1}$

The arithmetic mean for any n positive numbers $a_1, a_2, a_3, \ldots a_n$ is given by

$$\text{A.M.} = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

If a, A and b are in A.P., then A is called the arithmetic mean of numbers a and b and

i.e., $$A = \frac{a+b}{2}$$

If the terms of an A.P. are increased, decreased, multiplied or divided by the same constant, they still remain in A.P.

If $a_1, a_2, a_3 \ldots$ are in A.P. with common difference d, then

(i) $a_1 \pm k, a_2 \pm k, a_3 \pm k, \ldots$ are also in A.P with common difference d.

(ii) $a_1 k, a_2 k, a_3 k, \ldots$ are also in A.P with common difference dk $(k \neq 0)$.

and $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}$ ... are also in A.P. with common difference $\frac{d}{k}$ $(k \neq 0)$.

If $a_1, a_2, a_3 \ldots$ and $b_1, b_2, b_3 \ldots$ are two A.P., then

(i) $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$ are also in A.P

(ii) $a_1 b_1, a_2 b_2, a_3 b_3, \ldots$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ are not in A.P.

If $a_1, a_2, a_3 \ldots$ and a_n are in A.Ps, then

(i) $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$

(ii) $a_r = \frac{a_{r-k} + a_{r+k}}{2} \;\forall\; k, 0 \le k \le n - r$

(iii) If n^{th} term of any sequence is linear expression in n, then the sequence is an A.P.

(iv) If sum of n terms of any sequence is a quadratic expression in n, then sequence is an A.P.

9.1.2 ***A Geometric progression (G.P.)*** is a sequence in which each term except the first is obtained by multiplying the previous term by a non-zero constant called the **common ratio**. Let us consider a G.P. with first non-zero term a and common ratio r, i.e.,
$$a, ar, ar^2, \ldots, ar^{n-1}, \ldots$$

Here, common ratio $r = \dfrac{ar^{n-1}}{ar^{n-2}}$

The **general term** or ***n*th term** of G.P. is given by $a_n = ar^{n-1}$.

Last term l of a G.P. is same as the n^{th} term and is given by $l = ar^{n-1}$.

and the n^{th} term from the last is given by $a_n = \dfrac{l}{r^{n-1}}$

The sum S_n of the first n terms is given by

$$S_n = \frac{a(r^n - 1)}{r-1}, \qquad \text{if } r \neq 1$$

$$S_n = na \qquad \text{if } r = 1$$

If a, G and b are in G.P., then G is called the **geometric mean** of the numbers a and b and is given by

$$G = \sqrt{ab}$$

(i) If the terms of a G.P. are multiplied or divided by the same non-zero constant ($k \neq 0$), they still remain in G.P.

If $a_1, a_2, a_3, \ldots$, are in G.P., then $a_1 k, a_2 k, a_3 k, \ldots$ and $\dfrac{a_1}{k}, \dfrac{a_2}{k}, \dfrac{a_3}{k}, \ldots$ are also in G.P. with same common ratio, in particularly if $a_1, a_2, a_3, \ldots$ are in G.P., then

$\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}, \ldots$ are also in G.P.

(ii) If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two G.P.s, then $a_1 b_1, a_2 b_2, a_3 b_3, \ldots$ and $\dfrac{a_1}{b_1}, \dfrac{a_2}{b_2}, \dfrac{a_3}{b_3}, \ldots$ are also in G.P.

(iii) If $a_1, a_2, a_3, \ldots$ are in A.P. ($a_i > 0 \ \forall\ i$), then $x^{a_1}, x^{a_2}, x^{a_3}, \ldots$, are in G.P. ($\forall\ x > 0$)

(iv) If a_1, a_2, a_3, ..., a_n are in G.P., then $a_1\ a_n = a_2\ a_{n-1} = a_3\ a_{n-2} = ...$

9.1.3 *Important results on the sum of special sequences*

(i) Sum of the first n natural numbers:

$$\sum n = 1 + 2 + 3 + ... + n = \frac{n\,(n+1)}{2}$$

(ii) Sum of the squares of first n natural numbers.

$$\sum n^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n\,(n+1)\,(2n+1)}{6}$$

(iii) Sum of cubes of first n natural numbers:

$$\sum n^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n\,(n+1)}{2}\right]^2$$

9.2 Solved Examples

Short Answer Type

Example 1 The first term of an A.P. is a, the second term is b and the last term is c. Show that the sum of the A.P. is $\frac{(b+c-2a)\,(c+a)}{2\,(b-a)}$.

Solution Let d be the common diffrence and n be the number of terms of the A.P.

Since the first term is a and the second term is b

Therefore, $d = b - a$

Also, the last term is c, so

$$c = a + (n-1)\,(b - a) \quad (\text{since } d = b - a)$$

$$\Rightarrow \quad n - 1 = \frac{c-a}{b-a}$$

$$\Rightarrow \quad n = 1 + \frac{c-a}{b-a} = \frac{b-a+c-a}{b-a} = \frac{b+c-2a}{b-a}$$

Therefore, $$S_n = \frac{n}{2}(a+l) = \frac{(b+c-2a)}{2\,(b-a)}\,(a+c)$$

Example 2 The p^{th} term of an A.P. is a and q^{th} term is b. Prove that the sum of its $(p + q)$ terms is

$\dfrac{p+q}{2}\left[a+b+\dfrac{a-b}{p-q}\right]$.

Solution Let A be the first term and D be the common difference of the A.P. It is given that

$$t_p = a \Rightarrow A + (p-1)\,D = a \qquad ...\ (1)$$

$$t_q = b \Rightarrow A + (q-1)\,D = b \qquad ...\ (2)$$

Subtracting (2) from (1), we get

$$(p-1-q+1)\,D = a - b$$

$$\Rightarrow \qquad D = \frac{a-b}{p-q} \qquad ...\ (3)$$

Adding (1) and (2), we get

$$2A + (p+q-2)\,D = a + b$$

$$\Rightarrow \qquad 2A + (p+q-1)\,D = a + b + D$$

$$\Rightarrow \qquad 2A + (p+q-1)\,D = a + b + \frac{a-b}{p-q} \qquad ..\ (4)$$

Now

$$S_{p+q} = \frac{p+q}{2}\,[2A + (p+q-1)\,D]$$

$$= \frac{p+q}{2}\left[a+b+\frac{a-b}{p-q}\right]$$

[(using ... (3) and (4)]

Example 3 If there are $(2n + 1)$ terms in an A.P., then prove that the ratio of the sum of odd terms and the sum of even terms is $(n + 1) : n$

Solution Let a be the first term and d the common difference of the A.P. Also let S_1 be the sum of odd terms of A.P. having $(2n + 1)$ terms. Then

$$S_1 = a_1 + a_3 + a_5 + ... + a_{2n+1}$$

$$S_1 = \frac{n+1}{2}\,(a_1 + a_{2n+1})$$

$$S_1 = \frac{n+1}{2}\left[a + a + (2n+1-1)d\right]$$

$$= (n + 1)(a + nd)$$

Similarly, if S_2 denotes the sum of even terms, then

$$S_2 = \frac{n}{2}[2a + 2nd] = n(a + nd)$$

Hence
$$\frac{S_1}{S_2} = \frac{(n+1)(a+nd)}{n(a+nd)} = \frac{n+1}{n}$$

Example 4 At the end of each year the value of a certain machine has depreciated by 20% of its value at the beginning of that year. If its initial value was Rs 1250, find the value at the end of 5 years.

Solution After each year the value of the machine is 80% of its value the previous year so at the end of 5 years the machine will depreciate as many times as 5.

Hence, we have to find the 6th term of the G.P. whose first term a_1 is 1250 and common ratio r is .8.

Hence, value at the end 5 years = $t_6 = a_1 r^5 = 1250 (.8)^5 = 409.6$

Example 5 Find the sum of first 24 terms of the A.P. $a_1, a_2, a_3, \ldots$ if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.

Solution We know that in an A.P., the sum of the terms equidistant from the beginning and end is always the same and is equal to the sum of first and last term.

Therefore $\qquad d = b - a$

i.e., $\qquad a_1 + a_{24} = a_5 + a_{20} = a_{10} + a_{15}$

It is given that $(a_1 + a_{24}) + (a_5 + a_{20}) + (a_{10} + a_{15}) = 225$

$\Rightarrow (a_1 + a_{24}) + (a_1 + a_{24}) + (a_1 + a_{24}) = 225$

$\Rightarrow \qquad 3(a_1 + a_{24}) = 225$

$\Rightarrow \qquad a_1 + a_{24} = 75$

We know that $S_n = \frac{n}{2}[a + l]$, where a is the first term and l is the last term of an A.P.

Thus, $$S_{24} = \frac{24}{2}[a_1 + a_{24}] = 12 \times 75 = 900$$

Example 6 The product of three numbers in A.P. is 224, and the largest number is 7 times the smallest. Find the numbers.

Solution Let the three numbers in A.P. be $a - d, a, a + d$ $(d > 0)$

Now $\qquad (a - \text{d})\, a\, (a + \text{d}) = 224$

$\Rightarrow \qquad a\,(a^2 - d^2) = 224 \qquad \ldots (1)$

Now, since the largest number is 7 times the smallest, i.e., $a + d = 7\,(a - \text{d})$

Therefore, $\qquad d = \dfrac{3a}{4}$

Substituting this value of d in (1), we get

$$a\left(a^2 - \frac{9a^2}{16}\right) = 224$$

$$a = 8$$

and $\qquad d = \dfrac{3a}{4} = \dfrac{3}{4} \times 8 = 6$

Hence, the three numbers are 2, 8, 14.

Example 7 Show that $(x^2 + xy + y^2)$, $(z^2 + xz + x^2)$ and $(y^2 + yz + z^2)$ are consecutive terms of an A.P., if x, y and z are in A.P.

Solution The terms $(x^2 + xy + y^2)$, $(z^2 + xz + x^2)$ and $(y^2 + yz + z^2)$ will be in A.P. if

$$(z^2 + xz + x^2) - (x^2 + xy + y^2) = (y^2 + yz + z^2) - (z^2 + xz + x^2)$$

i.e., $\qquad z^2 + xz - xy - y^2 = y^2 + yz - xz - x^2$

i.e., $\qquad x^2 + z^2 + 2xz - y^2 = y^2 + yz + xy$

i.e., $\qquad (x + z)^2 - y^2 = y\,(x + y + z)$

i.e., $\qquad x + z - y = y$

i.e., $\qquad x + z = 2y$

which is true, since x, y, z are in A.P. Hence $x^2 + xy + y^2$, $z^2 + xz + x^2$, $y^2 + yz + z^2$ are in A.P.

Example 8 If a, b, c, d are in G.P., prove that $a^2 - b^2$, $b^2 - c^2$, $c^2 - d^2$ are also in G.P.

Solution Let r be the common ratio of the given G.P. Then

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = r$$

$\Rightarrow \qquad b = ar,\ c = br = ar^2,\ d = cr = ar^3$

Now, $\qquad a^2 - b^2 = a^2 - a^2r^2 = a^2\,(1 - r^2)$

$$b^2 - c^2 = a^2r^2 - a^2r^4 = a^2r^2\ (1 - r^2)$$

and $$c^2 - d^2 = a^2r^4 - a^2r^6 = a^2r^4\ (1 - r^2)$$

Therefore, $$\frac{b^2 - c^2}{a^2 - b^2} = \frac{c^2 - d^2}{b^2 - c^2} = r^2$$

Hence, $a^2 - b^2$, $b^2 - c^2$, $c^2 - d^2$ are in G.P.

Long Answer Type

Example 9 If the sum of m terms of an A.P. is equal to the sum of either the next n terms or the next p terms, then prove that

$$(m + n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m + p)\left(\frac{1}{m} - \frac{1}{n}\right)$$

Solution Let the A.P. be a, $a + d$, $a + 2d$, ... We are given

$$a_1 + a_2 + ... + a_m = a_{m+1} + a_{m+2} + ... + a_{m+n} \quad ... (1)$$

Adding $a_1 + a_2 + ... + a_m$ on both sides of (1), we get

$$2\ [a_1 + a_2 + ... + a_m] = a_1 + a_2 + ... + a_m + a_{m+1} + ... + a_{m+n}$$

$$2\ S_m = S_{m+n}$$

Therefore, $$2\frac{m}{2}\{2a + (m-1)d\} = \frac{m+n}{2}\{2a + (m+n-1)d\}$$

Putting $2a + (m - 1)\ d = x$ in the above equation, we get

$$mx = \frac{m+n}{2}\ (x + nd)$$

$$(2m - m - n)\ x = (m + n)\ nd$$

$$\Rightarrow \quad (m - n)\ x = (m + n)\ nd \quad ... (2)$$

Similarly, if $a_1 + a_2 + ... + a_m = a_{m+1} + a_{m+2} + ... + a_{m+p}$

Adding $a_1 + a_2 + ... + a_m$ on both sides

we get, $$2\ (a_1 + a_2 + ... + a_m) = a_1 + a_2 + ... + a_{m+1} + ... + a_{m+p}$$

or, $$2\ S_m = S_{m+p}$$

$$\Rightarrow \quad 2\left[\frac{m}{2}\{2a + (m-1)d\}\right] = \frac{m+p}{2}\ \{2a + (m + p - 1)d\}$$ which gives

i.e., $$(m - p)\ x = (m + p)pd \quad ... (3)$$

Dividing (2) by (3), we get

$$\frac{(m-n)x}{(m-p)x} = \frac{(m+n)nd}{(m+p)pd}$$

$$\Rightarrow \qquad (m-n)(m+p)p = (m-p)(m+n)n$$

Dividing both sides by mnp, we get

$$(m+p)\left(\frac{1}{n}-\frac{1}{m}\right) = (m+n)\left(\frac{1}{p}-\frac{1}{m}\right)$$

$$= (m+n)\left(\frac{1}{m}-\frac{1}{p}\right) = (m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$$

Example 10 If $a_1, a_2, ..., a_n$ are in A.P. with common difference d (where $d \neq 0$); then the sum of the series $\sin d$ (cosec a_1 cosec a_2 + cosec a_2 cosec a_3 + ...+ cosec a_{n-1} cosec a_n) is equal to cot a_1 – cot a_n

Solution We have

$\sin d$ (cosec a_1 cosec a_2 + cosec a_2 cosec a_3 + ...+ cosec a_{n-1} cosec a_n)

$$= \sin d\left[\frac{1}{\sin a_1 \sin a_2} + \frac{1}{\sin a_2 \sin a_3} + \ldots + \frac{1}{\sin a_{n-1} \sin a_n}\right]$$

$$= \frac{\sin(a_2-a_1)}{\sin a_1 \sin a_2} + \frac{\sin(a_3-a_2)}{\sin a_2 \sin a_3} + \ldots + \frac{\sin(a_n-a_{n-1})}{\sin a_{n-1} \sin a_n}$$

$$= \frac{\sin a_2 \cos a_1 - \cos a_2 \sin a_1)}{\sin a_1 \sin a_2} + \frac{\sin a_3 \cos a_2 - \cos a_3 \sin a_2)}{\sin a_2 \sin a_3} + \ldots + \frac{\sin a_n \cos a_{n-1} - \cos a_n \sin a_{n-1})}{\sin a_{n-1} \sin a_n}$$

$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3) + \ldots + (\cot a_{n-1} - \cot a_n)$

$= \cot a_1 - \cot a_n$

Example 11

(i) If a, b, c, d are four distinct positive quantities in A.P., then show that $bc > ad$

(ii) If a, b, c, d are four distinct positive quantities in G.P., then show that $a + d > b + c$

Solution

(i) Since a, b, c, d are in A.P., then A.M. > G.M., for the first three terms.

Therefore, $b > \sqrt{ac}$ $\qquad \left(\text{Here } \frac{a+c}{2} = b\right)$

Squaring, we get $\qquad b^2 > ac \qquad$ ... (1)

Similarly, for the last three terms

AM > GM

$c > \sqrt{bd}$ $\qquad \left(\text{Here } \frac{b+d}{2} = c\right)$

$c^2 > bd \qquad$ ... (2)

Multiplying (1) and (2), we get

$b^2c^2 > (ac)\ (bd)$

$\Rightarrow \quad bc > ad$

(ii) Since a, b, c, d are in G.P.

again A.M. > G.M. for the first three terms

$\frac{a+c}{2} > b$ $\qquad \left(\text{since } \sqrt{ac} = b\right)$

$\Rightarrow \quad a + c > 2b \qquad$ ... (3)

Similarly, for the last three terms

$\frac{b+d}{2} > c$ $\qquad \left(\text{since } \sqrt{bd} = c\right)$

$\Rightarrow \quad b + d > 2c \qquad$ ... (4)

Adding (3) and (4), we get

$(a + c) + (b + d) > 2b + 2c$

$a + d > b + c$

Eample 12 If a, b, c are three consecutive terms of an A.P. and x, y, z are three consecutive terms of a G.P. Then prove that

$x^{b-c} \cdot y^{c-a} \cdot z^{a-b} = 1$

Solution We have a, b, c as three consecutive terms of A.P. Then

$b - a = c - b = d$ $\qquad$ (say)

$c - a = 2d$

$a - b = -d$

Now $\quad x^{b-c} \cdot y^{c-a} \cdot z^{a-b} = x^{-d} \cdot y^{2d} \cdot z^{-d}$

$= x^{-d} (\sqrt{xz})^{2d} . z^{-d}$ (since $y = (\sqrt{xz})$) as x, y, z are G.P.)

$= x^{-d} . x^{d} . z^{d} . z^{-d}$

$= x^{-d+d} . z^{d-d}$

$= x^{\circ} z^{\circ} = 1$

Example 13 Find the natural number a for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function f satisfies $f(x + y) = f(x) . f(y)$ for all natural numbers x, y and further $f(1) = 2$.

Solution Given that

$$f(x + y) = f(x) . f(y) \text{ and } f(1) = 2$$

Therefore,

$$f(2) = f(1 + 1) = f(1) . f(1) = 2^2$$

$$f(3) = f(1 + 2) = f(1) . f(2) = 2^3$$

$$f(4) = f(1 + 3) = f(1) . f(3) = 2^4$$

and so on. Continuing the process, we obtain

$$f(k) = 2^k \text{ and } f(a) = 2^a$$

Hence

$$\sum_{k=1}^{n} f(a+k) = \sum_{k=1}^{n} f(a).f(k)$$

$$= f(a) \sum_{k=1}^{n} f(k)$$

$$= 2^a (2^1 + 2^2 + 2^3 + ... + 2^n)$$

$$= 2^a \left\{ \frac{2.\left(2^n - 1\right)}{2-1} \right\} = 2^{a+1} (2^n - 1) \qquad ... (1)$$

But, we are given

$$\sum_{k=1}^{n} f(a+k) = 16 (2^n - 1)$$

$$2^{a+1} (2^n - 1) = 16 (2^n - 1)$$

$\Rightarrow$ $\quad 2^{a+1} = 2^4 \Rightarrow a + 1 = 4$

$\Rightarrow$ $\quad a = 3$

Objective Type Questions

Choose the correct answer out of the four given options in Examples 14 to 23 (M.C.Q.).

Example 14 A sequence may be defined as a

(A) relation, whose range $\subseteq$ **N** (natural numbers)

(B) function whose range $\subseteq$ **N**

(C) function whose domain $\subseteq$ **N**

(D) progression having real values

Solution (C) is the correct answer. A sequence is a function $f : \mathbf{N} \to X$ having domain $\subseteq \mathbf{N}$

Example 15 If x, y, z are positive integers then value of expression $(x + y)(y + z)(z + x)$ is

(A) $= 8xyz$ (B) $> 8xyz$ (C) $< 8xyz$ (D) $= 4xyz$

Solution (B) is the correct answer, since

A.M. > GM., $\frac{x+y}{2} > \sqrt{xy}$, $\frac{y+z}{2} > \sqrt{yz}$ and $\frac{z+x}{2} > \sqrt{zx}$

Multiplying the three inequalities, we get

$$\frac{x+y}{2} \cdot \frac{y+z}{2} \cdot \frac{y+z}{2} > \sqrt{(xy)(yz)(zx)}$$

or, $(x + y)(y + z)(z + x) > 8\,xyz$

Example 16 In a G.P. of positive terms, if any term is equal to the sum of the next two terms. Then the common ratio of the G.P. is

(A) $\sin 18°$ (B) $2\cos 18°$ (C) $\cos 18°$ (D) $2\sin 18°$

Solution (D) is the correct answer, since

$$t_n = t_{n+1} + t_{n+2}$$

$\Rightarrow$ $ar^{n-1} = ar^n + ar^{n+1}$

$\Rightarrow$ $1 = r + r^2$

$$r = \frac{-1 \pm \sqrt{5}}{2}, \text{ since } r > 0$$

Therefore, $r = 2\frac{\sqrt{5}-1}{4} = 2\sin 18°$

Example 17 In an A.P. the pth term is q and the $(p+q)^{th}$ term is 0. Then the qth term is

(A) $-p$ (B p (C) $p+q$ (D) $p-q$

Solution (B) is the correct answer

Let a, d be the first term and common difference respectively.

Therefore, $T_p = a + (p-1)\, d = q$ and ... (1)

$T_{p+q} = a + (p+q-1)\, d = 0$ ... (2)

Subtracting (1), from (2) we get $qd = -q$

Substituting in (1) we get $a = q - (p-1)(-1) = q + p - 1$

Now $T_q = a + (q-1)\, d = q + p - 1 + (q-1)(-1)$

$= q + p - 1 - q + 1 = p$

Example 18 Let S be the sum, P be the product and R be the sum of the reciprocals of 3 terms of a G.P. Then $P^2 R^3 : S^3$ is equal to

(A) 1 : 1 (B) (common ratio)^n : 1

(C) (first term)^2 : (common ratio)^2 (D) none of these

Solution (A) is the correct answer

Let us take a G.P. with three terms $\frac{a}{r}, a, ar$. Then

$$S = \frac{a}{r} + a + ar = \frac{a(r^2+r+1)}{r}$$

$$P = a^3,\ R = \frac{r}{a} + \frac{1}{a} + \frac{1}{ar} = \frac{1}{a}\left(\frac{r^2+r+1}{r}\right)$$

$$\frac{P^2 R^3}{S^3} = \frac{a^6 \cdot \frac{1}{a^3}\left(\frac{r^2+r+1}{r}\right)^3}{a^3\left(\frac{r^2+r+1}{r}\right)^3} = 1$$

Therefore, the ratio is 1 : 1

Example 19 The 10th common term between the series

3 + 7 + 11 + ... and 1 + 6 + 11 + ... is

(A) 191 (B) 193 (C) 211 (D) None of these

Solution (A) is the correct answer.

The first common term is 11.

Now the next common term is obtained by adding L.C.M. of the common difference 4 and 5, i.e., 20.

Therefore, 10^{th} common term = T_{10} of the AP whose $a = 11$ and $d = 20$

$T_{10} = a + 9d = 11 + 9\,(20) = 191$

Example 20 In a G.P. of even number of terms, the sum of all terms is 5 times the sum of the odd terms. The common ratio of the G.P. is

(A) $\frac{-4}{5}$ (B) $\frac{1}{5}$ (C) 4 (D) none the these

Solution (C) is the correct answer

Let us consider a G.P. $a, ar, ar^2, \ldots$ with $2n$ terms. We have $\frac{a(r^{2n}-1)}{r-1} = \frac{5a\left((r^2)^n-1\right)}{r^2-1}$

(Since common ratio of odd terms will be r^2 and number of terms will be n)

$$\Rightarrow \quad \frac{a(r^{2n}-1)}{r-1} = 5\frac{a(r^{2n}-1)}{(r^2-1)}$$

$\Rightarrow a\,(r+1) = 5a$, i.e., $r = 4$

Example 21 The minimum value of the expression $3^x + 3^{1-x}$, $x \in \mathbf{R}$, is

(A) 0 (B) $\frac{1}{3}$ (C) 3 (D) $2\sqrt{3}$

Solution (D) is the correct answer.

We know A.M. ≥ G.M. for positive numbers.

Therefore, $\frac{3^x + 3^{1-x}}{2} \ge \sqrt{3^x \cdot 3^{1-x}}$

$$\Rightarrow \quad \frac{3^x + 3^{1-x}}{2} \ge \sqrt{3^x \cdot \frac{3}{3^x}}$$

$$\Rightarrow \quad 3^x + 3^{1-x} \ge 2\sqrt{3}$$

9.3 EXERCISE

Short Answer Type

1. The first term of an A.P.is a, and the sum of the first p terms is zero, show that the sum of its next q terms is $\dfrac{-a(p+q)q}{p-1}$. [**Hint:** Required sum = $S_{p+q} - S_p$]

2. A man saved Rs 66000 in 20 years. In each succeeding year after the first year he saved Rs 200 more than what he saved in the previous year. How much did he save in the first year?

3. A man accepts a position with an initial salary of Rs 5200 per month. It is understood that he will receive an automatic increase of Rs 320 in the very next month and each month thereafter.

   (a) Find his salary for the tenth month

   (b) What is his total earnings during the first year?

4. If the pth and qth terms of a G.P. are q and p respectively, show that its $(p+q)^{th}$ term is $\left(\dfrac{q^p}{p^q}\right)^{\frac{1}{p-q}}$.

5. A carpenter was hired to build 192 window frames. The first day he made five frames and each day, thereafter he made two more frames than he made the day before. How many days did it take him to finish the job?

6. We know the sum of the interior angles of a triangle is 180°. Show that the sums of the interior angles of polygons with 3, 4, 5, 6, ... sides form an arithmetic progression. Find the sum of the interior angles for a 21 sided polygon.

7. A side of an equilateral triangle is 20cm long. A second equilateral triangle is inscribed in it by joining the mid points of the sides of the first triangle. The process is continued as shown in the accompanying diagram. Find the perimeter of the sixth inscribed equilateral triangle.

8. In a potato race 20 potatoes are placed in a line at intervals of 4 metres with the first potato 24 metres from the starting point. A contestant is required to bring the potatoes back to the starting place one at a time. How far would he run in bringing back all the potatoes?

9. In a cricket tournament 16 school teams participated. A sum of Rs 8000 is to be awarded among themselves as prize money. If the last placed team is awarded

Rs 275 in prize money and the award increases by the same amount for successive finishing places, how much amount will the first place team receive?

10. If $a_1, a_2, a_3, ..., a_n$ are in A.P., where $a_i > 0$ for all i, show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+...+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

11. Find the sum of the series

$(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + ...$ to (i) n terms (ii) 10 terms

12. Find the r^{th} term of an A.P. sum of whose first n terms is $2n + 3n^2$.

[**Hint:** $a_n = S_n - S_{n-1}$]

Long Answer Type

13. If A is the arithmetic mean and G_1, G_2 be two geometric means between any two numbers, then prove that

$$2A = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}$$

14. If $\theta_1, \theta_2, \theta_3, ..., \theta_n$ are in A.P., whose common difference is d, show that

$$\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + ... + \sec\theta_{n-1} \sec\theta_n = \frac{\tan\theta_n - \tan\theta_1}{\sin d}.$$

15. If the sum of p terms of an A.P. is q and the sum of q terms is p, show that the sum of $p + q$ terms is $-(p + q)$. Also, find the sum of first $p - q$ terms ($p > q$).

16. If p^{th}, q^{th}, and r^{th} terms of an A.P. and G.P. are both a, b and c respectively, show that

$$a^{b-c} \cdot b^{c-a} \cdot c^{a-b} = 1$$

Objective Type Questions

Choose the correct answer out of the four given options in each of the Exercises 17 to 26 (M.C.Q.).

17. If the sum of n terms of an A.P. is given by

$S_n = 3n + 2n^2$, then the common difference of the A.P. is

(A) 3 (B) 2 (C) 6 (D) 4

18. The third term of G.P. is 4. The product of its first 5 terms is

(A) 4^3 (B) 4^4 (C) 4^5 (D) None of these

19. If 9 times the 9th term of an A.P. is equal to 13 times the 13th term, then the 22nd term of the A.P. is

(A) 0 (B) 22 (C) 220 (D) 198

20. If x, $2y$, $3z$ are in A.P., where the distinct numbers x, y, z are in G.P. then the common ratio of the G.P. is

(A) 3 (B) $\frac{1}{3}$ (C) 2 (D) $\frac{1}{2}$

21. If in an A.P., $S_n = q\,n^2$ and $S_m = qm^2$, where S_r denotes the sum of r terms of the A.P., then S_q equals

(A) $\frac{q^3}{2}$ (B) mnq (C) q^3 (D) $(m + n)\,q^2$

22. Let S_n denote the sum of the first n terms of an A.P. If $S_{2n} = 3S_n$ then $S_{3n} : S_n$ is equal to

(A) 4 (B) 6 (C) 8 (D) 10

23. The minimum value of $4^x + 4^{1-x}$, $x \in$ R, is

(A) 2 (B) 4 (C) 1 (D) 0

24. Let S_n denote the sum of the cubes of the first n natural numbers and s_n denote the sum of the first n natural numbers. Then $\sum_{r=1}^{n} \frac{S_r}{s_r}$ equals

(A) $\frac{n(n+1)(n+2)}{6}$ (B) $\frac{n(n+1)}{2}$

(C) $\frac{n^2+3n+2}{2}$ (D) None of these

25. If t_n denotes the nth term of the series 2 + 3 + 6 + 11 + 18 + ... then t_{50} is

(A) $49^2 - 1$ (B) 49^2 (C) $50^2 + 1$ (D) $49^2 + 2$

26. The lengths of three unequal edges of a rectangular solid block are in G.P. The volume of the block is 216 cm^3 and the total surface area is 252cm^2. The length of the longest edge is

(A) 12 cm (B) 6 cm (C) 18 cm (D) 3 cm

Fill in the blanks in the Exercises 27 to 29.

27. For a, b, c to be in G.P. the value of $\dfrac{a-b}{b-c}$ is equal to ....

28. The sum of terms equidistant from the beginning and end in an A.P. is equal to ....

29. The third term of a G.P. is 4, the product of the first five terms is ....

State whether statement in Exercises 30 to 34 are True or False.

30. Two sequences cannot be in both A.P. and G.P. together.

31. Every progression is a sequence but the converse, i.e., every sequence is also a progression need not necessarily be true.

32. Any term of an A.P. (except first) is equal to half the sum of terms which are equidistant from it.

33. The sum or difference of two G.P.s, is again a G.P.

34. If the sum of n terms of a sequence is quadratic expression then it always represents an A.P.

Match the questions given under Column I with their appropriate answers given under the Column II.

35.

	Column I		**Column II**
(a)	$4, 1, \frac{1}{4}, \frac{1}{16}$	(i)	A.P.
(b)	2, 3, 5, 7	(ii)	sequence
(c)	13, 8, 3, –2, –7	(iii)	G.P.

36.

	Column I		**Column II**
(a)	$1^2 + 2^2 + 3^2 + ...+n^2$	(i)	$\left(\frac{n(n+1)}{2}\right)^2$
(b)	$1^3 + 2^3 + 3^3 + ...+n^3$	(ii)	$n\ (n + 1)$
(c)	$2 + 4 + 6 + ... + 2n$	(iii)	$\frac{n(n+1)(2n+1)}{6}$
(d)	$1 + 2 + 3 +...+ n$	(iv)	$\frac{n(n+1)}{2}$